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Canonical quantization of so-called non-Lagrangian systems

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Abstract. We present an approach to the canonical quantization of systems with equations of motion that are historically called non-Lagrangian equations. Our viewpoint of this problem is the following: despite the fact that a set of differential equations cannot be directly identified with a set of Euler–Lagrange equations, one can reformulate such a set in an equivalent first-order form that can always be treated as the Euler–Lagrange equations of a certain action. We construct such an action explicitly. It turns out that in the general case the hamiltonization and canonical quantization of such an action are non-trivial problems, since the theory involves time-dependent constraints. We adopt the general approach of hamiltonization and canonical quantization for such theories as described in D.M. Gitman, I.V. Tyutin, Quantization of Fields with Constraints (Springer, Berlin, 1990). to the case under consideration. There exists an ambiguity (that cannot be reduced to the addition of a total time derivative) in associating a Lagrange function with a given set of equations. We present a complete description of this ambiguity. The proposed scheme is applied to the quantization of a general quadratic theory. In addition, we consider the quantization of a damped oscillator and of a radiating point-like charge.

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1 Introduction

It is well known that some physical systems, like dissipative systems [1], the Dirac monopole [2], etc., are usually described in terms of second-order equations of motion which cannot be directly identified with the Euler–Lagrange equations for an action principle. Following traditional terminology, we call such equations of motion non-Lagrangian equations in what follows. Sometimes (but not always) non-Lagrangian equations can be reduced to Euler–Lagrange equations by multiplying by the so-called integrating multiplier [3–5]. The existence of an action principle for a given physical system, or, what is the same, the existence of a Lagrange function for such a system, allows one to proceed with canonical quantization schemes. This, in particular, stresses the importance of formulating an action principle for any physical system.

In the present work we discuss an approach to the construction of quantum theories that in the classical limit reproduce non-Lagrangian equations of motion for the mean values. In fact, we consider the canonical quantization of Lagrangian theories with time-dependent constraints that are related to the non-Lagrangian systems. To this end, on the classical level, we reduce the non-Lagrangian equations of motion to an equivalent set of first-order differential equations. For such equations one can always construct an action principle; the corresponding consideration is presented in Sect. 2 and, partially, is based on results of [8–10]. The hamiltonization of the constructed Lagrangian theory leads to a Hamiltonian theory with time-dependent constraints, as demonstrated in Sect. 3. Thus, we show that systems traditionally called non-Lagrangian are, in fact, equivalent to certain first-order Lagrangian systems; however, with time-dependent constraints in the Hamiltonian formulation. The canonical quantization of the latter theory is not a trivial problem (this follows from the general consideration of [14]) and is presented in Sect. 4. It is known that, on the classical level, there exists an ambiguity in constructing the Lagrange function (which is not reduced to a total time derivative) for a given set of equations [8–13]. We completely describe such an ambiguity for the case under consideration. We apply the general approach to formulate the canonical quantization in the case of theories with arbitrary linear inhomogeneous equations of motion (general quadratic theories); see Sect. 5. Next we consider the canonical quantization of a damped harmonic oscillator (Sect. 6) and of a radiating point-like charge (Sect. 7).

2 Action principle for non-Lagrangian systems

Let a system with n degrees of freedom be described by a set of n non-Lagrangian second-order differential equa-

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tions of motion. To construct an action principle, we replace these equations by an equivalent set of $2n$ first-order differential equations, solvable with respect to the time derivatives (which is always possible). Suppose such a set has the form

$$\dot{x}^\alpha = f^\alpha(t,x)\,, \quad \alpha = 1,\dots,2n\,, \tag{1}$$

where $f^\alpha(t,x)$ are functions of the indicated arguments, and by the dots we denote time derivatives of the coordinates. Since these equations are of first order, the action $S[x]$ that yields (1) as Euler–Lagrange equations must be linear in the first time derivative, $\dot{x}^\alpha$. Its general form is

$$S[x] = \int \mathrm{d}t L\,, \quad L = J_\alpha \dot{x}^\alpha - H\,, \tag{2}$$

where $J_\alpha = J_\alpha(t,x)$ and $H = H(t,x)$ are certain functions of the indicated arguments. The Euler–Lagrange equations corresponding to (2) are

$$\begin{aligned}\frac{\delta S}{\delta x} = \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0 \Longrightarrow \\ -\partial_\alpha H - \partial_t J_\alpha + (\partial_\alpha J_\beta - \partial_\beta J_\alpha)\,\dot{x}^\beta = 0\,,\end{aligned} \tag{3}$$

where the following notation is used:

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha}\,, \quad \partial_t = \frac{\partial}{\partial t}\,.$$

Denoting the combination $(\partial_\alpha J_\beta - \partial_\beta J_\alpha)$ by $\Omega_{\alpha\beta}$,

$$\Omega_{\alpha\beta} = \partial_\alpha J_\beta - \partial_\beta J_\alpha = \Omega_{\alpha\beta}(t,x) = -\Omega_{\beta\alpha}(t,x)\,, \tag{4}$$

we rewrite (3) as follows:

$$\Omega_{\alpha\beta}\dot{x}^\beta = \partial_\alpha H + \partial_t J_\alpha\,. \tag{5}$$

Equations (3) or (5) can be identified with (1), provided

$$\det \Omega_{\alpha\beta} \neq 0\,, \tag{6}$$

$$\Omega_{\alpha\beta} f^\beta - \partial_t J_\alpha = \partial_\alpha H\,. \tag{7}$$

The functions J_α and H can be found from the conditions (4)–(7) if the matrix $\Omega_{\alpha\beta}$ is given. Assuming that J_α and H are smooth functions, the consistency condition for (7) implies

$$\begin{aligned}\partial_\beta\left(\Omega_{\alpha\gamma} f^\gamma\right) - \partial_\alpha\left(\Omega_{\beta\gamma} f^\gamma\right) + \partial_t \Omega_{\alpha\beta} = 0 \Longrightarrow \\ \partial_t \Omega_{\alpha\beta} + \mathcal{L}_f \Omega_{\alpha\beta} = 0\,,\end{aligned} \tag{8}$$

where $\mathcal{L}_f \Omega_{\alpha\beta}$ is the Lie derivative of $\Omega_{\alpha\beta}$ along the vector field f^γ. In addition, one can verify that the matrix $\Omega_{\alpha\beta}$, see (4), obeys the Jacobi identity ($\Omega_{\alpha\beta}$ is a symplectic matrix):

$$\partial_\alpha \Omega_{\beta\gamma} + \partial_\beta \Omega_{\gamma\alpha} + \partial_\gamma \Omega_{\alpha\beta} = 0\,. \tag{9}$$

Now we are going to analyze these equations. It is known that the general solution $\Omega_{\alpha\beta}$ of (8) can be constructed with the help of a solution of the Cauchy problem for (1). Suppose that such a solution is known; let

$$x^\alpha = \varphi^\alpha(t, x_{(0)})\,, \quad x^\alpha_{(0)} = \varphi^\alpha(0, x_{(0)}) \tag{10}$$

be a solution of (1) for any $x_{(0)} = \left(x^\alpha_{(0)}\right)$, and let $\chi^\alpha(t,x)$ be the inverse function with respect to $\varphi^\alpha(t, x_{(0)})$, i.e.,

$$\begin{aligned}x^\alpha = \varphi^\alpha(t, x_{(0)}) \Longrightarrow x^\alpha_{(0)} = \chi^\alpha(t,x)\,, \\ x^\alpha \equiv \varphi^\alpha(t, \chi^\alpha)\,, \quad \partial_\alpha \chi^\gamma|_{t=0} = \delta^\alpha_\gamma\,.\end{aligned} \tag{11}$$

Then

$$\Omega_{\alpha\beta}(t,x) = \partial_\alpha \chi^\gamma \Omega^{(0)}_{\gamma\delta}(\chi) \partial_\beta \chi^\delta\,, \tag{12}$$

where the matrix $\Omega^{(0)}_{\alpha\beta}$ is the initial condition for $\Omega_{\alpha\beta}$,

$$\Omega_{\alpha\beta}(t,x)|_{t=0} = \Omega^{(0)}_{\alpha\beta}(x)\,.$$

It follows from (9) at $t=0$ that the matrix $\Omega^{(0)}_{\alpha\beta}(x)$ obeys the Jacobi identity, so that the general structure of this matrix is (we do not consider global problems which arise from a non-trivial topology of the x^α-space)

$$\Omega^{(0)}_{\alpha\beta} = \partial_\alpha j_\beta - \partial_\beta j_\alpha\,, \tag{13}$$

where the $j_\alpha(x)$ are certain arbitrary functions. Then (12) implies

$$\Omega_{\alpha\beta} = \partial_\alpha \psi_\beta - \partial_\beta \psi_\alpha\,, \psi_\alpha(t,x) = j_\beta\left(\chi(t,x)\right) \partial_\alpha \chi^\beta(t,x)\,. \tag{14}$$

On the other hand, (4) must hold,

$$\partial_\alpha \psi_\beta - \partial_\beta \psi_\alpha = \partial_\alpha J_\beta - \partial_\beta J_\alpha\,,$$

which implies that

$$J_\alpha(t,x) = \psi_\alpha + \partial_\alpha \varphi = j_\beta\left(\chi(t,x)\right) \partial_\alpha \chi^\beta(t,x) + \partial_\alpha \varphi(t,x)\,, \tag{15}$$

where $\varphi(t,x)$ is an arbitrary function. One can represent $J_\alpha(t,x)$ in another form, for which the ambiguity related to the arbitrary functions $j_\beta(x)$ is incorporated in the matrix $\Omega^{(0)}_{\alpha\beta}$. To this end, we recall that the general solution for $J_\alpha(t,x)$ of (4), provided that $\Omega_{\alpha\beta}$ is a given antisymmetric matrix that obeys the Jacobi identity, is given by

$$J_\alpha(t,x) = \int_0^1 x^\beta \Omega_{\beta\alpha}(t,sx) s\,\mathrm{d}s + \partial_\alpha \varphi(t,x)\,, \tag{16}$$

where $\varphi(x)$ is an arbitrary function. Substituting (12) into (16), we obtain

$$\begin{aligned}J_\alpha(t,y) = \int_0^1 y^\beta \left[\partial_\alpha \chi^\gamma \Omega^{(0)}_{\gamma\delta}(\chi)\,\partial_\beta \chi^\delta\right]\Big|_{x=sy} s\,\mathrm{d}s \\ + \partial_\alpha \varphi(t,y)\,.\end{aligned} \tag{17}$$

Equation (15) or (17) describes the whole ambiguity (arbitrary functions $j_\beta(x)$ and $\varphi(t,x)$, or arbitrary symplectic matrix $\Omega^{(0)}_{\gamma\delta}$ and arbitrary function $\varphi(t,x)$) in constructing the term $J_\alpha(t,x)$ of the Lagrange function (2).

One can also see that choosing the matrix $\Omega^{(0)}_{\alpha\beta}(x)$ to be non-singular, we guarantee the non-singularity (condition (6)) for the matrix $\Omega_{\alpha\beta}(t,x)$, since the components of the latter are given by a change of variables (12).

To restore the term H in the Lagrange function (2), we need to solve (7) with respect to H. To this end, we remind the reader that the general solution of the equation $\partial_i f = g_i$, provided that the vector g_i is a gradient, is given by

$$f(x) = \int_0^1 \mathrm{d}s x^i g_i(sx) + c\,,$$

where c is a constant. Taking the above into account, we obtain for H the following representation:

$$H(t,x) = \int_0^1 \mathrm{d}s x^\beta \left[\Omega_{\beta\alpha}(t,sx) f^\alpha(t,sx) - \partial_t J_\beta(t,sx)\right] + c(t)\,, \tag{18}$$

where $c(t)$ is an arbitrary function of time, and $\Omega_{\beta\alpha}$ and J_β are given by (12) and (17), respectively. All the arbitrariness in constructing H is thus due to the arbitrary symplectic matrix $\Omega^{(0)}_{\gamma\delta}$, due to the arbitrary functions $\varphi(t,x)$ entering into $\Omega_{\beta\alpha}$ and J_β and due to $c(t)$.

We see that there exist a family of actions (2) that lead to the same equations of motion (1). It is easy to see that actions with the same $\Omega^{(0)}_{\gamma\delta}$ but different functions $\varphi(t,x)$ and $c(t)$ differ by a total time derivative (we call such a difference trivial). A difference in the Lagrange functions related to a different choice of the symplectic matrices $\Omega^{(0)}_{\alpha\beta}$ is not trivial. The corresponding Lagrangians are known as s-equivalent Lagrangians. In spite of the fact that actions with a non-trivial difference lead to the same equations of motion, they lead in general to different Hamiltonian formulations and to different quantum theories in the process of quantization. However, any quantum theory that is obtained by the quantization procedure developed below obeys the correspondence principle; i.e., in the classical limit, the equations of motion for the mean values coincide with (1). The equations of motion (1) do not contain any additional information that can be used in choosing a "right" quantum theory. Only physical considerations or a comparison with experiment may be used for this aim. Below, we are going to consider hamiltonization and subsequent quantization of the action (2) with the following choice of the symplectic matrix $\Omega^{(0)}_{\gamma\delta}$:

$$\Omega^{(0)}_{\alpha\beta} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{19}$$

where $\mathbf{I}$ is an $n\times n$ unit matrix, and $\mathbf{0}$ denotes an $n\times n$ zero matrix. This choice of the action leads to the canonical commutation relations for the original variables on the quantum level. Hamiltonization and quantization of the action (2) with different choices of the symplectic matrix $\Omega^{(0)}_{\gamma\delta}$ can be fulfilled in the same manner, but they technically look more clumsy.

Note that (19) implies that there exist only two possibilities for the matrix Ω in (12). Namely, it is either a canonical symplectic matrix, which is possible only if the initial equations (1) are canonical Hamiltonian equations (or, equivalently, Lagrangian equations of motion for the first-order action), or it must depend on time, which is the case of non-Lagrangian equations.

The first-order action (2) can be regarded as a Lagrangian action or as a Hamiltonian action with non-canonical Poisson brackets. An equivalent second-order Lagrangian formulation is always possible; however, it may include additional variables [15].

One should remark that it is always possible to construct a Lagrangian action for non-Lagrangian second-order equations in an extended configuration space following a simple idea first proposed by Bateman [22]. Such a Lagrangian has the form of a sum of initial equations of motion that are multiplied by the corresponding Lagrangian multipliers – these are new variables. The Euler–Lagrange equations for such an action contain, besides the initial equations, some new equations of motion for the Lagrange multipliers. In such an approach one has to think of how to interpret the new variables already on the classical level. Additional difficulties (an indefinite metric) can appear in the course of the quantization.

As an example, we consider a theory with equations of motion of the form[1]

$$\dot{x} = A(t)x + j(t)\,. \tag{20}$$

We call such a theory a general quadratic theory. Let us apply the above considerations to the construction of the action principle for such a theory.

The solution of the Cauchy problem for (20) reads

$$x(t) = \Gamma(t)x_{(0)} + \gamma(t)\,, \tag{21}$$

where the matrix $\Gamma(t)$ is the fundamental solution of (20), i.e.,

$$\dot{\Gamma} = A\Gamma\,, \quad \Gamma(0) = 1\,, \tag{22}$$

and $\gamma(t)$ is a partial solution of (20). Then, following (12), we construct the matrix Ω,

$$\Omega = \Lambda^{\mathrm{T}}\Omega^{(0)}\Lambda\,, \quad \Lambda = \Gamma^{-1}\,, \tag{23}$$

and we find the functions J and H according to (17) and (18),

$$J = \frac{1}{2}x\Omega\,, \quad H = \frac{1}{2}xBx - Cx\,, \tag{24}$$

where

$$B = \frac{1}{2}\left(\Omega A - A^{\mathrm{T}}\Omega\right), \quad C = \Omega j\,. \tag{25}$$

[1] Here we use matrix notation, $x = (x^\alpha)$, $A(t) = \left(A(t)^\alpha_\beta\right)$, $j(t) = (j(t)^\alpha)$, $\alpha,\beta = 1,\ldots,2n$.

Thus, the action functional for the general quadratic theory is

$$S[x] = \frac{1}{2}\int \mathrm{d}t\,(x\Omega\dot{x} - xBx - 2Cx)\ . \tag{26}$$

Another approach to constructing the action functional for the general quadratic theory was proposed in [1].

Note that Darboux coordinates x_0 can be written by means of a matrix Λ as follows:

$$x \to x_0 = R^{-1}(t)\Lambda(t)x\,. \tag{27}$$

Here, $R(t)$ is an arbitrary matrix of a linear (generally time-dependent) canonical transformation:

$$R^{\mathrm{T}}(t)\Omega^{(0)}R(t) = \Omega^{(0)}\,.$$

In terms of the coordinates x_0, the action (26) takes the form

$$S[x] = \frac{1}{2}\int \mathrm{d}t\left(x_0\Omega^{(0)}\dot{x}_0 + x_0R^{\mathrm{T}}\Omega^{(0)}\dot{R}x_0 - 2C\Gamma Rx_0\right)\ . \tag{28}$$

The Darboux coordinates (27) can be divided into coordinates and the corresponding momenta. The Euler–Lagrange equations for the action (28) have the form of canonical Hamilton equations with the Hamiltonian

$$H_0 = -\frac{1}{2}x_0R^{\mathrm{T}}\Omega^{(0)}\dot{R}x_0 + C\Gamma Rx_0\,. \tag{29}$$

Note that the choice $R = \mathrm{const}$ yields a trivial Hamiltonian, which is consistent with the fact that in this case the x_0 are the initial data without dynamics.

3 Hamiltonian formulation

We are now going to consider the action (2) as a Lagrangian action with the Lagrange function

$$L = J_\alpha\,(t,x)\,\dot{x}^\alpha - H\,(t,x) \tag{30}$$

and construct the corresponding Hamiltonian formulation. To this end, we follow the general[2] scheme of [14]. We first construct the action $S^v[x,\pi,v]$, which, in this case, has the form

$$S^v[x,\pi,v] = \int \left[J_\alpha\,(t,x)\,v^\alpha - H\,(t,x) + \pi_\alpha\,(\dot{x}^\alpha - v^\alpha)\right]\mathrm{d}t \tag{31}$$

and depends on the momenta π_α conjugate to the coordinates x^α, as well as on the velocities v^α. The equations

$$\frac{\delta S^v}{\delta v^\alpha} = \Phi_\alpha\,(t,x,\pi) = \pi_\alpha - J_\alpha(t,x) = 0 \tag{32}$$

do not allow one to express the velocities by x and π, which implies the appearance of primary constraints Φ_α, and the velocities v^α become Lagrangian multipliers to these constraints, so that the action (31) becomes a Hamiltonian action of a theory with the primary constraints (32),

$$\begin{aligned} S_{\mathrm{H}} &= \int \mathrm{d}t\left\{\pi_\alpha\dot{x}^\alpha - H^{(1)}\right\} \\ H^{(1)} &= H(t,x) + \lambda^\alpha\Phi_\alpha(t,x,\pi)\,, \end{aligned} \tag{33}$$

with the equations of motion

$$\dot{\eta} = \left\{\eta, H^{(1)}\right\}\,, \quad \Phi = 0\,, \tag{34}$$

where $\eta = (x,\pi)$.

The primary constraints are second-class ones. Indeed, we have, in virtue of (6),

$$\{\Phi_\alpha,\Phi_\beta\} = \Omega_{\alpha\beta}(t,x) \Longrightarrow \det\{\Phi_\alpha,\Phi_\beta\} \neq 0\,. \tag{35}$$

Thus, secondary constraints do not appear, and all λ are determined by the consistency conditions for the primary constraints:

$$\begin{aligned} &\dot{\Phi}_\alpha = \partial_t\Phi_\alpha + \{\Phi_\alpha, H^{(1)}\} = 0 \\ &\quad\Longrightarrow -\partial_tJ_\alpha - \partial_\alpha H + \lambda^\beta\{\Phi_\alpha,\Phi_\beta\} = 0 \\ &\quad\Longrightarrow \lambda^\beta = \omega^{\beta\alpha}\left(\partial_tJ_\alpha + \partial_\alpha H\right)\,, \quad \omega^{\beta\alpha} = \Omega^{-1}_{\beta\alpha}\,. \end{aligned} \tag{36}$$

Using the Lagrange multipliers (36) in (34), we can write these equations in the form

$$\dot{\eta} = \{\eta, H\}_{D(\Phi)} + \{\eta,\Phi_\alpha\}\omega^{\alpha\beta}\partial_tJ_\beta\,, \quad \Phi = 0\,, \tag{37}$$

where $\{\cdots,\cdots\}_{D(\Phi)}$ are the Dirac brackets with respect to the second-class constraints Φ. For the canonical variables, the Dirac brackets are

$$\begin{aligned} &\{x^\alpha, x^\beta\}_{D(\Phi)} = \omega^{\alpha\beta}\,, \\ &\{\pi_\alpha,\pi_\beta\}_{D(\Phi)} = \partial_\alpha J_\rho\omega^{\rho\gamma}\partial_\beta J_\gamma\,, \\ &\{x^\alpha,\pi_\beta\}_{D(\Phi)} = \delta^\alpha_\beta + \omega^{\alpha\gamma}\partial_\beta J_\gamma\,. \end{aligned} \tag{38}$$

Formally introducing a momentum ϵ conjugate to the time t, and defining the Poisson brackets in an extended space of the canonical variables $(x,\pi;t,\epsilon) = (\eta;t,\epsilon)$ [14], we can rewrite (37) as follows:

$$\dot{\eta} = \{\eta, H+\epsilon\}_{D(\Phi)}\,, \quad \Phi = 0\,. \tag{39}$$

Equations (39) represent a Hamiltonian formulation of non-Lagrangian systems with the first-order equations of motion (1). We note that the Hamiltonian constraints in this formulation are second-class ones and depend on time explicitly. The canonical quantization of theories with time-dependent second-class constraints can be carried out

[2] Note that some of the J_α may be equal to zero; for instance, if one deals with a canonical Hamiltonian action. In this case, one obtains the constraints $\Phi_\alpha = \pi_\alpha = 0$. Another way to examine this case is to use the method of hamiltonization for theories with degenerate coordinates [16].

along the lines of [14]. Below, we present the details of such a quantization, and then adapt it to the system under consideration.

4 Canonical quantization

For a Hamiltonian theory with time-dependent second-class constraints, the quantization procedure in the "Schrödinger" picture is realized as follows. To the phase-space variables η of a theory with time-dependent second-class constraints $\Phi_l(\eta,t)$ are assigned operators $\hat\eta(t)$ subject to the equal-time commutation relations and the constraints' equations:

$$[\hat\eta^A(t),\hat\eta^B(t)] = \mathrm{i}\{\eta^A,\eta^B\}_{D(\Phi)}|_{\eta=\hat\eta}\,, \quad \Phi_l(\hat\eta(t),t)=0\,. \tag{40}$$

Their time evolution is postulated to be (we neglect the problem of operator ordering [17])

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat\eta(t) = \{\eta,\epsilon\}_{D(\Phi)}|_{\eta=\hat\eta} = -\{\eta,\Phi_l\}\{\Phi,\Phi\}^{-1}_{ll'}\partial_t\Phi_{l'}|_{\eta=\hat\eta}\,. \tag{41}$$

To each physical quantity F, given in the Hamiltonian formulation by a function $F(t,\eta)$, we assign a "Schrödinger" operator $\hat F(t)$, by the rule $\hat F(t)=F(t,\hat\eta(t))$. For arbitrary "Schrödinger" operators $\hat F(t)$ and $\hat G(t)$, the relation

$$[\hat F(t),\hat G(t)] = \mathrm{i}\{F,G\}_{D(\Phi)}|_{\eta=\hat\eta} \tag{42}$$

holds as a consequence of (40). The quantum states of the system are described by the vectors Ψ of a Hilbert space with a scalar product (Ψ,Ψ'). Their time evolution is determined by the Schrödinger equation,

$$\mathrm{i}\frac{\partial\Psi(t)}{\partial t} = \hat H\Psi(t)\,, \tag{43}$$

where the quantum Hamiltonian $\hat H$ is constructed according to the classical function $H(t,\eta)$ as $\hat H(t)=H(t,\hat\eta(t))$. The mean values $\langle F\rangle_t$ of a physical quantity F are determined as the mean values of a corresponding "Schrödinger" operator $\hat F(t)=F(t,\hat\eta(t))$ with respect to the state vectors $\Psi(t)$,

$$\langle F\rangle_t = (\Psi(t),\hat F(t)\Psi(t))\,. \tag{44}$$

Provided that $\hat H$ is a self-adjoint operator, the time evolution of the state vectors $\Psi(t)$ is unitary,

$$\Psi(t)=U(t)\Psi(0)\,, \quad U^+(t)=U^{-1}(t)\,, \tag{45}$$

where $U(t)$ is the evolution operator.

In the Heisenberg picture, where the state vectors are "frozen" and the time evolution is governed by the Heisenberg operators $\check\eta(t)=U^{-1}(t)\hat\eta(t)U(t)$, one can see [14] that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\check\eta &= \{\eta,H(t,\eta)+\epsilon\}_{D(\Phi)}|_{\eta=\check\eta}\,,\\ [\check\eta^A(t),\check\eta^B(t)] &= \mathrm{i}\{\eta^A,\eta^B\}_{D(\Phi)}|_{\eta=\check\eta}\,, \quad \Phi(\check\eta(t),t)=0\,,\end{aligned} \tag{46}$$

while for Heisenberg operators $\check F(t)=U^{-1}(t)\hat F(t)U(t)=F(t,\check\eta(t))$, and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\check F(t) = \{F(t,\eta),H(t,\eta)+\epsilon\}_{D(\Phi)}|_{\eta=\check\eta}\,, \tag{47}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\check F(t) = -\mathrm{i}\left[\check F(t),\check H(t)\right] + \{F(t,\eta),\epsilon\}_{D(\Phi)}|_{\eta=\check\eta}\,. \tag{48}$$

The mean values $\langle F\rangle_t$ in the Heisenberg picture in accord with (44) and (45) are determined by

$$\langle F\rangle_t = \left(\Psi(0),\check F(t)\Psi(0)\right)\,. \tag{49}$$

The above quantization provides the fulfillment of the correspondence principle because the quantum equations (46) have the same form as the classical ones, (39).

Note that the time dependence of the Heisenberg operators in the theories under consideration is not unitary in the general case. In other words, there exists no ("Hamiltonian") operator whose commutator with a physical quantity can produce its total time derivative. This is explained by the existence of two factors that determine the time evolution of a Heisenberg operator. The first one is the unitary evolution of the state vector in the "Schrödinger" picture, while the second one is the time variation of a "Schrödinger" operators $\hat\eta$, which in general has a non-unitary character. The existence of these two factors is related to the division of the right-hand side of (48) into two summands. Physically, this is explained by the fact that the dynamics develops on a surface which changes with time – in the general case, in a non-unitary way.

Below, we apply the above quantization scheme to the system under consideration. Taking into account the Dirac brackets (38), we can write the equal-time commutation relations (40) for phase-space operators as follows:

$$\begin{aligned}\left[\hat x^\alpha,\hat x^\beta\right] &= \mathrm{i}\omega^{\alpha\beta}|_{x=\hat x}\,,\\ [\hat\pi_\alpha,\hat\pi_\beta] &= \mathrm{i}\;\partial_\alpha J_\rho\omega^{\rho\gamma}\partial_\beta J_\gamma\big|_{x=\hat x}\,,\\ [\hat x^\alpha,\hat\pi_\beta] &= \mathrm{i}\delta^\alpha_\beta + \mathrm{i}\;\omega^{\alpha\gamma}\partial_\beta J_\gamma|_{x=\hat x}\,.\end{aligned} \tag{50}$$

In this case, the classical Hamiltonian H does not depend on the momenta π_α, and therefore, in order to determine the quantum Hamiltonian $\hat H$ we need to know only the time dependence of the operators $\hat x^\alpha$. From (41) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat x^\alpha = \omega^{\alpha\beta}(t,x)\partial_t J_\beta(t,x)\big|_{x=\hat x}\,. \tag{51}$$

5 Quantization of general quadratic theory

The quantum-mechanical description of quadratic systems is a widely discussed physical problem, which has a number of important applications; see, e.g., [18–21] and references therein. Almost all of these references deal with the case of

"Hamiltonian" quadratic systems, i.e., systems described by canonical Hamiltonian equations of motion. On the other hand, we consider a general quadratic system, i.e., a system described by the arbitrary linear inhomogeneous equations of motion (20). In this case the conditions (50) and (51) become

$$\left[\hat{x}^\alpha, \hat{x}^\beta\right] = \mathrm{i}\omega^{\alpha\beta}(t)\,, \tag{52}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}^\alpha = -\frac{1}{2}\omega^{\alpha\beta}(t)\dot{\Omega}_{\beta\gamma}(t)\hat{x}^\gamma\,. \tag{53}$$

The time dependence of the operators $\hat{x}$ can easily be found:

$$\hat{x}^\alpha(t) = \Phi^{\alpha\beta}(t)\hat{x}_0^\beta\,. \tag{54}$$

Here, the matrix Φ obeys the equation

$$\dot{\Phi} = -\frac{1}{2}\omega\dot{\Omega}\Phi\,, \quad \Phi(0) = E\,, \tag{55}$$

and the operators $\hat{x}_0$ obey the following commutation relations:

$$\left[\hat{x}_0^\alpha, \hat{x}_0^\beta\right] = \mathrm{i}\left(\Omega^{(0)}_{\alpha\beta}\right)^{-1} = \mathrm{i}\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & \mathbf{0}\end{pmatrix}; \tag{56}$$

see (19). In what follows, it is useful to divide the operators $\hat{x}_0^\alpha$ into the operators of coordinates proper and the corresponding momenta: $\hat{x}_0^\alpha = (\hat{q}^i, \hat{p}_i)$, $\alpha = 1, \dots, 2n$, $i = 1, \dots, n$. The operators $\hat{q}$ and $\hat{p}$ obey the canonical commutation relations

$$\left[\hat{q}^i, \hat{p}_j\right] = \mathrm{i}\delta^i_j\,, \quad \left[\hat{q}^i, \hat{q}^j\right] = [\hat{p}_i, \hat{p}_j] = 0\,. \tag{57}$$

The quantum Hamiltonian in (43) takes the form

$$\hat{H} = \frac{1}{2}\hat{x}_0\Phi^{\mathrm{T}}B\Phi\hat{x}_0 - C\Phi\hat{x}_0\,, \tag{58}$$

where the matrix B is determined by (25).

The above quantization is equivalent to quantization in Darboux coordinates, and the transformation $x \to \Phi(t)x_0$ provides, by itself, a passage to the Darboux coordinates x_0, because (55) implies

$$\Phi^{\mathrm{T}}\Omega\Phi = \Omega_0\,. \tag{59}$$

Namely, in the coordinates x_0 the Poisson bracket is canonical. Therefore, $\Phi = \Gamma(t)R(t)$, where $\Gamma(t)$ is a fundamental solution of the system (20). However, in contrast to the classical theory, now the matrix $R(t)$ is fixed; it must obey the conditions

$$\dot{R} = \Omega_0\Gamma^{\mathrm{T}}B\Gamma R\,, \quad R(0) = E\,. \tag{60}$$

Thus, using (29) one can also rewrite the Hamiltonian in (58) as follows:

$$\hat{H} = -\frac{1}{2}\hat{x}_0R^{\mathrm{T}}\Omega^{(0)}\dot{R}\hat{x}_0 + C\Gamma R\hat{x}_0\,. \tag{61}$$

It is remarkable that, if the matrix A that determines the set of equations (20) is constant, the matrix that determines the quadratic part of the Hamiltonian in (58) is constant as well and equals

$$\Phi^{\mathrm{T}}B\Phi = B(0) = \frac{1}{2}\left(\Omega^{(0)}A - A^{\mathrm{T}}\Omega^{(0)}\right)\,. \tag{62}$$

This fact is easy to observe, because the time derivative of this matrix, in view of (55), (23) and (25), is equal to zero:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^{\mathrm{T}}B\Phi) = 0\,.$$

Thus, in this case, as distinct from the general case, the matrix Φ can be determined from the set of algebraic equations (59) and (62).

Note that if we start from a canonical Hamiltonian system the above quantization coincides with the usual canonical quantization, because in this case equation (53) becomes $\mathrm{d}\hat{x}/\mathrm{d}t = 0$, i.e., $\hat{x}(t) = \hat{x}_0$.

In the Heisenberg picture, the equations (46) for the operators $\check{x}$ take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\check{x} = A(t)\check{x} + j(t)\,, \tag{63}$$

$$[\check{x}^\alpha, \check{x}^\beta] = \mathrm{i}\omega^{\alpha\beta}(t)\,. \tag{64}$$

Equations (63) coincide with the classical equations of motion (20) (the correspondence principle); however, the commutation relations (64) differ from the canonical ones. So, the evolution of the operators $\check{x}$ can be written as

$$\check{x}(t) = \Gamma(t)\check{x}_0 + \gamma(t)\,, \tag{65}$$

where the operators $\check{x}_0$ as well as $\hat{x}_0$ obey the canonical commutation relations

$$\left[\check{x}_0^\alpha, \check{x}_0^\beta\right] = \mathrm{i}\left(\Omega^{(0)}_{\alpha\beta}\right)^{-1} = \mathrm{i}\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & \mathbf{0}\end{pmatrix}\,. \tag{66}$$

Thus, the mean values $\langle F\rangle_t$ of a physical quantity F according to (49) are determined as the mean values of the corresponding operator $\check{F}(t) = F(t, \check{x}(t))$ with respect to the initial states vectors $\Psi(0)$, i.e.

$$\langle F\rangle_t = \left(\Psi(0), F\left(t, \Gamma(t)\check{x}_0 + \gamma(t)\right)\Psi(0)\right)\,. \tag{67}$$

We see that the quantum evolution of physical quantities in general quadratic systems is completely determined by the classical one.

6 Quantization of a damped harmonic oscillator

The above formulated quantization of non-Lagrangian theories and, in particular, of general quadratic theories can be immediately applied to the quantization of a damped harmonic oscillator. The latter problem has attracted attention for already more than 50 years. There exist different approaches to its solution, but none of them seems to

be the final version that does not contain weak points; see e.g. [1, 23–44].

The classical equation of motion for a damped harmonic oscillator is non-Lagrangian; it has the form

$$\ddot{r} + 2\alpha\dot{r} + \omega^2 r = 0\,, \tag{68}$$

where ω is the angular frequency and $\alpha \geq 0$ is the friction coefficient. Introducing an auxiliary variable $y = \dot{r}$, we reduce (68) to the following equivalent pair of first-order equations:

$$\dot{r} = y\,, \quad \dot{y} = -\omega^2 r - 2\alpha y\,. \tag{69}$$

In the way proposed in Sect. 2, we construct an action S that implies (69) as the Euler–Lagrange equations,

$$S = \frac{1}{2}\int \mathrm{d}t\left[y\dot{r} - r\dot{y} - (y^2 + 2\alpha r y + \omega^2 r^2)\right]\mathrm{e}^{2\alpha t}\,. \tag{70}$$

Note that (68) can be represented as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\alpha t}\dot{r}\right) + \mathrm{e}^{2\alpha t}\omega^2 r = 0\,,$$

i.e., as a Lagrangian equation of motion with time-dependent mass and frequency. In this case, the mass $\mathrm{e}^{2\alpha t}$ is nothing but an integrating multiplier for (68); however, as already mentioned, an integrating multiplier does not always exist [3, 10].

Next, we proceed with the canonical quantization described in the previous section. The equal-time commutation relations (52) and (53), determining the time evolution of the "Schrödinger" operators $\hat{r}$ and $\hat{y}$, are

$$[\hat{r}, \hat{y}] = \mathrm{i}\mathrm{e}^{-2\alpha t}\,, \quad [\hat{r}, \hat{r}] = [\hat{y}, \hat{y}] = 0\,, \tag{71}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{r} = -\alpha\hat{r}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\hat{y} = -\alpha\hat{y}\,. \tag{72}$$

A solution of these equations has the form

$$\hat{r} = \mathrm{e}^{-\alpha t}\hat{q}\,, \quad \hat{y} = \mathrm{e}^{-\alpha t}\hat{p}\,, \tag{73}$$

where the operators $\hat{q}$ and $\hat{p}$ obey the canonical commutation relations

$$[\hat{q}, \hat{p}] = \mathrm{i}\,, \quad [\hat{q}, \hat{q}] = [\hat{p}, \hat{p}] = 0\,.$$

According to (58), the corresponding quantum Hamiltonian reads

$$\hat{H} = \frac{1}{2}\left[\hat{p}^2 + \alpha\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right) + \omega^2\hat{q}^2\right]\,.$$

It can be modified to the form

$$\hat{H} = \frac{1}{2}\left[\hat{P}^2 + \left(\omega^2 - \alpha^2\right)\hat{Q}^2\right] \tag{74}$$

with the help of the canonical transformation $(\hat{p}, \hat{q}) \to (\hat{P}, \hat{Q})$, where $\hat{P} = \hat{p} + \alpha\hat{q}$ and $\hat{Q} = \hat{q}$. The corresponding generating function is $W = qP - \alpha q^2/2$.

As usual we define the classical energy of the system by

$$E = \frac{1}{2}\left(\dot{r}^2 + \omega^2 r^2\right) = \frac{1}{2}\left(y^2 + \omega^2 r^2\right)\,.$$

One can easily see that the energy depends on time as follows: $E = E_0\mathrm{e}^{-2\alpha t}$. Using (73), we obtain an expression for the operator $\hat{E}$ that corresponds to the classical quantity E,

$$\hat{E} = \frac{1}{2}\mathrm{e}^{-2\alpha t}\left[\hat{P}^2 - \alpha(\hat{P}\hat{Q} + \hat{Q}\hat{P}) + (\omega^2 + \alpha^2)\hat{Q}^2\right]\,. \tag{75}$$

Let us consider the underdamped case, $\alpha < \omega$. Then (74) is the Hamiltonian of a harmonic oscillator with an angular frequency $\tilde{\omega} = \sqrt{\omega^2 - \alpha^2}$. Stationary states of the corresponding Schrödinger equation have the form

$$\Psi_n = \mathrm{e}^{-\mathrm{i}E_n t}\psi_n(Q)\,, \quad E_n = \tilde{\omega}\left(n + \frac{1}{2}\right)\,, \quad n = 0, 1, \dots\,,$$

$$\psi_n(Q) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\tilde{\omega}}{\pi}\right)^{1/4}\mathrm{e}^{-\tilde{\omega}Q^2/2}H_n\left(Q\sqrt{\tilde{\omega}}\right)\,. \tag{76}$$

The mean values of the energy (of the operator (75)) in such states can easily be calculated:

$$\langle E\rangle_n = \left(n + \frac{1}{2}\right)\frac{\omega^2}{\tilde{\omega}}\mathrm{e}^{-2\alpha t}\,. \tag{77}$$

At each fixed time instant, the energy spectrum is discrete; however, it decreases with time exactly as in the classical theory. The same conclusion was derived in [37, 39] where a second-order action obtained by the integrating–multiplier method was taken as the starting point for the quantization. Quantization of the damped oscillator following Bateman (see above) meets with serious difficulties, such as an indefinite metric, etc. [39, 40].

Overdamped cases, when $\alpha \geq \omega$, correspond to aperiodic motion in classical theory [45]. Its quantum interpretation is not clear due to the continuous character of the Hamiltonian spectrum.

A non-trivial generalization of (68) could be an n-dimensional damped oscillator:[3]

$$\ddot{r} + 2\mathbf{a}\dot{r} + \omega r = 0\,, \tag{78}$$

with the matrices $\mathbf{a}$ and ω being constant and symmetric. Introducing auxiliary variables, $y = \dot{r}$, $y = (y^i)$, we reduce (78) to the following set of first-order equations:

$$\dot{x} = Ax\,, \quad (x = (x^\alpha) = (r^i, y^i))\,, \tag{79}$$

$$A = \begin{pmatrix} 0 & \mathbf{I} \\ -\omega & -2\mathbf{a} \end{pmatrix}\,.$$

This is a set of linear equations with constant coefficients. In this case, the corresponding quantum Hamiltonian can be constructed, according to (62), as follows:

$$\hat{H} = \frac{1}{2}\hat{x}_0 B(0)\hat{x}_0\,, \tag{80}$$

[3] Here, we use matrix notation, $r = (r^i)$, $\mathbf{a} = \left(a^i_j\right)$, $\omega = \left(\omega^i_j\right)$, $i, j = 1, \dots, n$.

where

$$B(0)=\begin{pmatrix}\omega & \mathbf{a}\\ \mathbf{a} & \mathbf{I}\end{pmatrix},$$

and the operators $\hat{x}_0$ can be divided into the operators of the coordinates proper, $\hat{q}$, and those of the corresponding momenta, $\hat{p}$, with the canonical commutation relations (57). The further solution of the quantum problem with quadratic Hamiltonian (80) may follow, for example, [18].

7 Quantization of radiating point-like charge

The equations of motion for a non-relativistic particle moving in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{H}$ with account taken of the back reaction of the radiation emitted by the particle have the form [46]

$$\begin{aligned} &m\ddot{\mathbf{r}}=\mathbf{F}+\mathbf{f}\quad(\mathbf{r}=(x,y,z))\,,\\ &\mathbf{F}=e\mathbf{E}+\frac{\mathrm{e}}{c}[\dot{\mathbf{r}}\times\mathbf{H}]\,,\quad \mathbf{f}=\frac{2e^2}{3c^3}\dddot{\mathbf{r}}\,. \end{aligned}\tag{81}$$

Here $\mathbf{F}$ is the Lorentz force, $\mathbf{f}$ is the force of the back reaction of the radiation, e is the charge of the particle, and c is the velocity of light. Derivatives with respect to time are denoted by dots.

These equations are of third order; therefore, the trajectory of a charged particle cannot be uniquely specified only by the initial position and velocity of the particle. It has also been pointed out that, together with physically meaningful solutions, the equations (81) have a set of non-physical solutions [46]. However, in the case when the back reaction force $\mathbf{f}$ is small compared to the Lorentz force $\mathbf{F}$,

$$|\mathbf{f}|\ll|\mathbf{F}|\,,\tag{82}$$

these equations can be reduced to second-order equations by means of a reduction of the order procedure. Then the above mentioned problem with non-physical solutions does not appear. In the reduction procedure, equations (81) are replaced by second-order equations, $\ddot{\mathbf{r}}=\mathbf{g}(\mathbf{r},\dot{\mathbf{r}},e)$, such that all the solutions of the latter equations would be solutions of (81). The latter requirement implies a partial differential equation on the function $\mathbf{g}(\mathbf{r},\dot{\mathbf{r}},e)$ having a unique solution with the natural condition $\mathbf{g}(\mathbf{r},\dot{\mathbf{r}},0)=0$; see e.g. [10, 46].

Consider, for example, the particular case $\mathbf{E}=0$, $\mathbf{H}=(0,0,H=\mathrm{const})$. In such a case, the reduced second-order equations have the form [10]

$$\begin{aligned} &\ddot{x}=-\alpha\dot{x}-\beta\dot{y}\,,\quad \ddot{y}=\beta\dot{x}-\alpha\dot{y}\,,\quad \ddot{z}=0\,,\\ &\alpha=\frac{\sqrt{6}\sqrt{3+\sqrt{9+64e^6H^2}}-6}{8e^2}\approx\frac{2}{3}e^4H^2\,,\\ &\beta=\frac{eH\sqrt{6}}{\sqrt{3+\sqrt{9+64e^6H^2}}}\approx eH\,. \end{aligned}\tag{83}$$

Here we have set $m=c=1$ for simplicity. Since the evolution along the z-axis represents the free motion and decouples from the dynamics in the xy-plane, we restrict our consideration to the first two equations. At $\alpha=0$, equations (83) are Lorentz equations with an "effective" magnetic field $\beta=(0,0,\beta/e)$. In this case, the trajectories are concentric circles. If $\alpha\neq0$, the particle spirals to the origin of xy-plane. Therefore, it is natural to treat α as a friction coefficient.

In order to construct an action functional for the non-Lagrangian second-order equations (83), we introduce new variables as follows:

$$p=\dot{x}+\frac{\beta}{2}y\,,\quad q=\dot{y}-\frac{\beta}{2}x\,.$$

In the new variables, we have a set of first-order equations,

$$\begin{aligned} &\dot{x}=p-\frac{\beta}{2}y\,,\quad \dot{y}=q+\frac{\beta}{2}x\,,\\ &\dot{p}=-\frac{\beta}{2}q-\frac{\beta^2}{4}x-\alpha\left(p-\frac{\beta}{2}y\right),\\ &\dot{q}=\frac{\beta}{2}p-\frac{\beta^2}{4}y-\alpha\left(q+\frac{\beta}{2}x\right). \end{aligned}\tag{84}$$

According to the general formulas (24), (25), and (26), we construct the following action for the set (84):

$$\begin{aligned} S=\frac{1}{2(\alpha^2+\beta^2)}\int \mathrm{e}^{\alpha t}\big[&a(p\dot{x}-x\dot{p}+q\dot{y}-y\dot{q})\\ &+\beta(q\dot{x}-x\dot{q}+y\dot{p}-p\dot{y})\\ &+c(p\dot{q}-q\dot{p})+d(x\dot{y}-y\dot{x})\\ &-e(p^2+q^2)-f(x^2+y^2)-g(px+qy)\\ &-j(qx-py)\big]\,\mathrm{d}t\,, \end{aligned}\tag{85}$$

where the time-dependent functions a,b,c,d,e,f,g, and j are

$$\begin{aligned} &a=\alpha^2\cos(\beta t)+\beta^2\cosh(-\alpha t)\,,\\ &b=\alpha^2\sin(\beta t)+\alpha\beta\mathrm{e}^{-\alpha t}-\alpha\beta\cos(\beta t)\,,\\ &c=e^{-\alpha t}\beta-2\alpha\sin(\beta t),\\ &d=-\frac{\beta^3}{2}\sinh(-\alpha t)-\frac{1}{2}\alpha\beta\sin(\beta t)+\alpha^2\beta\left[\cos(\beta t)-\mathrm{e}^{-\alpha t}\right],\\ &e=\mathrm{e}^{-\alpha t}\beta^2+\alpha^2\cos(\beta t)-\alpha\beta\sin(\beta t)\,,\\ &f=\frac{\beta}{4}\left\{\beta^3\mathrm{e}^{\alpha t}+\beta\alpha^2\cos(\beta t)+\alpha[\beta^2+2\alpha^2]\sin(\beta t)\right\},\\ &g=\alpha\beta^2\cos(\beta t)+\alpha^3\cos(\beta t)\,,\\ &j=\alpha^3\sin(\beta t)+e^{\alpha t}\beta^3+\alpha^2\beta\cos(\beta t)\,. \end{aligned}$$

In the limit of zero friction, $\alpha\to0$, this action is reduced to the usual action for a charged particle in a homogeneous magnetic field β.

The set (84) is linear with constant coefficients. For such a case, the corresponding quantum Hamiltonian can be constructed according to (62) as follows:

$$\begin{aligned} \hat{H}=\frac{1}{2}\Big[&\hat{p}_1^2+\hat{p}_2^2+\frac{\alpha}{2}\left(\hat{p}_1\hat{q}_1+\hat{q}_1\hat{p}_1+\hat{p}_2\hat{q}_2+\hat{q}_2\hat{p}_2\right)\\ &+\beta(\hat{p}_2\hat{q}_1-\hat{p}_1\hat{q}_2)+\frac{\beta^2}{4}\left(\hat{q}_1^2+\hat{q}_2^2\right)\Big]\,, \end{aligned}\tag{86}$$

with the operators $\hat{q}_i$ and $\hat{p}_j$ obeying the canonical commutation relations,

$$[\hat{q}_i, \hat{p}_j] = \mathrm{i}\delta_{ij}, \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0\,, \quad i,j = 1,2\,.$$

By a canonical transformation, $(\hat{p}_1, \hat{q}_1; \hat{p}_2, \hat{q}_2) \to (\hat{p}, \hat{x}; \hat{q}, \hat{y})$, where

$$\hat{p} = \hat{p}_1 + \frac{\alpha}{2}\hat{q}_1,\ \hat{x} = \hat{q}_1,\ \hat{q} = \hat{p}_2 + \frac{\alpha}{2}\hat{q}_2,\ \hat{y} = \hat{q}_2\,,$$

we reduce (86) to the form

$$\hat{H} = \frac{1}{2}\left[\hat{p}^2 + \hat{q}^2 + \beta(\hat{q}\hat{x} - \hat{p}\hat{y}) + \frac{\beta^2 - \alpha^2}{4}\left(\hat{x}^2 + \hat{y}^2\right)\right]. \tag{87}$$

The condition (82) in the case under consideration implies $\alpha \ll \beta$, which is why α^2 will be omitted in (87) in what follows.

Consider the eigenstates Ψ for two mutually commuting operators $\hat{H}$ and $\hat{L} = \hat{p}\hat{y} - \hat{q}\hat{x}$,

$$\hat{H}\Psi = E\Psi\,,\ \hat{L}\Psi = M\Psi\,. \tag{88}$$

It is convenient to perform the following canonical transformation: $(\hat{p}, \hat{x}; \hat{q}, \hat{y}) \to (\hat{P}, \hat{X}; \hat{Q}, \hat{Y})$,

$$\hat{P} = \hat{p} - \frac{\beta}{2}\hat{y}\,, \quad \hat{X} = \frac{1}{\beta}\left(\hat{q} + \frac{\beta}{2}\hat{x}\right),$$
$$\hat{Q} = \hat{q} - \frac{\beta}{2}\hat{x}\,, \quad \hat{Y} = \frac{1}{\beta}\left(\hat{p} + \frac{\beta}{2}\hat{y}\right).$$

It is easy to see that

$$\hat{H} = \frac{1}{2}\left(\hat{P}^2 + \beta^2\hat{X}^2\right), \quad \hat{L} = \beta^{-1}(\hat{H}_1 - \hat{H})\,,$$
$$\hat{H}_1 = \frac{1}{2}\left(\hat{Q}^2 + \beta^2\hat{Y}^2\right).$$

The operators $\hat{H}$ and $\hat{H}_1$ are the Hamiltonians of two independent harmonic oscillators. Next, we can separate the variables, solving (88). Thus, we obtain the solution of the eigenvalue problem (88),

$$\Psi = \Psi_{n,l}(X,Y) = \psi_n(X)\psi_l(Y)\,,$$
$$E_n = \beta\left(n + \frac{1}{2}\right), \quad M_{nl} = l - n\,, \quad n,l = 0,1,2,\dots\,,$$

where ψ_n and ψ_l are eigenstates of the Hamiltonians $\hat{H}$ and $\hat{H}_1$, respectively (given e.g. by (76)). Finally, the stationary states $\Psi(t)$ of the corresponding Schrödinger equation with the Hamiltonian $\hat{H}$ have the form

$$\Psi(X,Y,t) = \mathrm{e}^{-\mathrm{i}E_n t}\Psi_{n,l}(X,Y)\,. \tag{89}$$

We define the classical energy E of the system under consideration according to [45] as the mechanical energy of the system without friction,

$$E = \frac{1}{2}\left[p^2 + q^2 + \beta(qx - py) + \frac{\beta^2}{4}\left(x^2 + y^2\right)\right].$$

One can see that the energy depends on time as follows: $E = E_0\mathrm{e}^{-2\alpha t}$. The operator $\hat{E}$ that corresponds to the classical quantity E reads

$$\hat{E} = \frac{1}{2}\left[\hat{P}^2 + \beta^2\hat{X}^2 + \alpha\left(\hat{X}\hat{Y} - \hat{P}\hat{Q}\right)\right]\mathrm{e}^{-2\alpha t}$$
$$- 2\alpha\left(\frac{\alpha}{\beta}\right)\left[\hat{P}\hat{Y} + \hat{Q}\hat{X}\right]\mathrm{e}^{-2\alpha t} + o\left(\frac{\alpha}{\beta}\right).$$

The mean values of this operator in the stationary states (89) can easily be calculated; they are

$$\langle E\rangle_{nl} = \beta\left(n + \frac{1}{2}\right)\mathrm{e}^{-2\alpha t}\,.$$

Similar to the case of a damped oscillator considered above, at each fixed instant, the energy spectrum is discrete; however, it decreases with time, exactly as in classical theory.

We would like to note that in [10] it was shown that, although an action principle for the second-order equations (83) describing a radiating point-like charge does exist, none of the possible corresponding Lagrangians in the limit of $\alpha \to 0$ reduces to the Lagrangian of a particle in a magnetic field modulo a total time derivative. That is, in the case of a radiating point-like charge, a perturbation (in the friction parameter α) of the second-order action does not correspond to a perturbation of the equations of motion (83). For this reason, we expect some difficulties with the limit of $\alpha \to 0$ in the quantum theory of a radiating point-like charge resulting from quantization based on an action functional in the second-order form (such a quantization for the damped harmonic oscillator was presented in [38, 39]).

8 Concluding remarks

We stress that any non-degenerate set of differential equations written in an equivalent first-order form can be derived from an action principle. In the general case, such a set does not provide enough information to fix a class of quantum theories that, in the classical limit, provide this set of differential equations for the mean values. Therefore, physical considerations must be used to choose an adequate quantum theory. In particular, if one definitely knows that a non-Lagrangian set of equations describes a dissipative system, which is subjected to dissipation due to an essential interaction with the environment (reservoir), it is reasonable to consider the system and the reservoir as two interacting subsystems of a closed system. Then the quantum description of the dissipative subsystem can be obtained from the quantum theory of the whole system by averaging over the reservoir. Such an approach was developed in many articles [24–26, 28–34]. However, one cannot consider such an approach as a quantization of the initial dissipative subsystem, since quantization has already been done for the whole system. In the present article, we consider an approach in which we actually quantize the system with a given set of equations. It turns out that

its "non-Lagrangian" behavior is due to a time-dependent external field. It is a principally different physical situation in comparison with dissipation of a subsystem. However, quantum theories obtained from our procedure may be useful for describing some quantum-mechanical properties of both dissipative systems and "non-Lagrangian" systems of another physical nature, like a monopole.

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